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A Nonlinear Implicit Code for Relativistic Electron Beam Tracking Studies

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The formulation of an implicit code is described. The code solves the three-dimensional field equations without linearization. This code is applied to the study of beam tracking. Some applications are suggested.

I. INTRODUCTION

Computational models that are linearized about an axisymmetric equilibrium have proven invaluable in the study of instabilities of propagating beams. For example, linearized (monopole/dipole) particle simulation codes [1] as well as linearized codes with simplified models of beam dynamics [2] have been used for hose instability analysis. However, there are other areas of beam propagation phenomenology in which large departure from axisymmetry is essential and inevitable, and in which such linearized models can be quite misleading. For example, in the presence of transverse external magnetic or electric fields, different "slices" of the beam (defined by the distance ζ from the beam head) may be subject to different accelerations $F/(m\gamma)$, where F is the force and γ, which may be a function of ζ, is the relativistic factor. In such situations, the beam may tear apart, or alternatively it may reach a non-axisymmetric equilibrium in which its cohesive self-forces counterbalance the sheared external forces. As another example, we have recently shown that there are important situations in which the beam head is in a grossly unstable equilibrium (unstable to simple zero-frequency transverse displacements) when it is propagating on-axis in a density or conductivity channel, but in which another and more stable equilibrium exists with the beam head off-axis by a prescribed amount. Thus axisymmetry is broken and the beam tends to track the channel by riding down the channel walls, rather than the channel center. Moreover, the equilibrium displacement from the channel axis increases from the head of the beam to its tail, increasing the departure from axisymmetry.

To treat problems such as these, it is necessary to develop a fully three-dimensional solution of the electromagnetic field equations; monopole/dipole approximations will not do. In dealing with very highly relativistic paraxial beams we shall assume,

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however, that it is sufficient to solve the three-dimensional form of Lee's reduced field equations [3], rather then the complete Maxwell equations. This paper deals mainly with the numerical methods which we have developed to solve the field equations. To date, we have only addressed the problem of determining non-axisymmetric beam equilibria, in which the beam is specified by a prescribed density profile $n_b(\mathbf{r} - \mathbf{Y}(\zeta), \zeta)$ [4]. Each slice of the beam is allowed transverse (noninfinitesimal) displacement $Y(\zeta)$, but is not allowed to distort. No particle dynamics and no time dependences have been studied to date. Eventually we shall have to study the stability of these equilibria to internal distortions and to dynamic modes such as hose, but this can only be done after the correct equilibria have been determined, a task which is by no means trivial either computationally, mathematically or physically.

The code we have developed solves the full set of Lee field equations without linearization. The radial dependence is solved by finite difference with nonuniform grids. The theta dependence is solved by fast Fourier transform. The advantage of this method is that the theta derivatives are more than second-order accurate [5]. An explicit scheme will not do because of severe time step limitation. This can be seen from the following: the Lee field equations are

$$\frac{\partial A}{\partial \zeta} = \eta \nabla_\perp^2 (A + \phi) + \eta J_b,$$

$$\nabla_\perp^2 \frac{\partial A}{\partial \zeta} = (\nabla \sigma) \cdot (\nabla \phi) + \sigma \nabla_\perp^2 \phi,$$

where $\eta \equiv \sigma^{-1}$. Both η and σ behave like diffusion coefficients in the equations. Near the head of the beam, the resistivity η is very large, but near the tail, the conductivity σ is very large. The difference can easily be three orders of magnitude. So any explicit scheme that takes care of the solution at one end will have severe time step limitations at the other end.

This code involves the solution of a bi-tridiagonal system with complex matrix coefficients. The complexity in calculation is offset by the fact that for the first time one is able to find the self-consistent fields and the force acting on the beam from head to tail for arbitrary beam displacement.

We will not present the physics results investigated here; they will be reported in later publications. Instead, we shall concentrate on the numerical formulation of the code. In Section II, we review briefly the convolution sum, and in Section III we apply it to a simple diffusion equation. In Section IV, the method of solving the field equations is described. In Section V, we discuss some applications. In Appendix A, we describe one way of inverting a bi-tridiagonal system with complex matrix coefficients. In Appendix B, we describe some experience in running a code of this type.

II. Review of Convolution Coding

Let $A(\theta)$ and $B(\theta)$ be two functions that depend on θ. The expansion of the product of A and B in a finite Fourier series is

$$A(\theta)\,B(\theta) = \sum_{l=-N}^{N} A_l e^{-il\theta} \sum_{m=-N}^{N} B_m e^{-im\theta}$$
$$= \sum_{l=-N}^{N} \sum_{m=-N}^{N} A_l B_m e^{-i(l+m)\theta}. \tag{1}$$

Let $n = l + m$, then (1) becomes

$$AB = \sum_{m=-N}^{N} \sum_{n=m-N}^{m+N} A_{n-m} B_m e^{-in\theta}. \tag{2}$$

We want to convert (2) to a form like $\sum_{n=-N}^{N}(\ )\,e^{-in\theta}$. Breaking the double sum to two sums and interchanging the direction of integration, we obtain

$$AB = \left\{ \left[\sum_{n=-2N}^{0} \sum_{m=-N}^{n+N} \right] + \left[\sum_{n=1}^{2N} \sum_{m=n-N}^{N} \right] \right\} A_{n-m} B_m e^{-in\theta}. \tag{3}$$

Equation (3) can be approximated by dropping the higher-order modes, i.e., we replace $\sum_{n=-2N}^{2N}$ by $\sum_{n=-N}^{N}$. This can be justified, provided that the magnitude of a transform goes down rapidly with higher mode number. Equation (3) becomes

$$A(\theta)\,B(\theta) \approx \left\{ \left[\sum_{n=-N}^{0} \sum_{m=-N}^{n+N} \right] + \left[\sum_{n=1}^{N} \sum_{m=n-N}^{N} \right] \right\} A_{n-m} B_m e^{-in\theta}. \tag{4}$$

III. Implicit Formulation of a Diffusion Equation by the Transform Method

The RHS of a diffusion equation

$$\frac{\partial A}{\partial \zeta}(r, \theta) = \eta(r, \theta)\, L[A(r, \theta)] \tag{5}$$

with $L \equiv \nabla_{\perp}^2$, can be expressed as

$$\text{RHS} = \left\{ \left[\sum_{n=-N}^{0} \sum_{m=-N}^{n+N} \right] + \left[\sum_{n=1}^{N} \sum_{m=n-N}^{N} \right] \right\} \eta_{n-m} (LA)_m e^{-in\theta}. \tag{6}$$

In matrix form, (6) becomes (dropping $e^{-in\theta}$)

$$\begin{array}{llllll} \eta_0(LA)_{-N} + & \cdots & + \eta_{-N}(LA)_0 & + 0 & & \\ \eta_1(LA)_{-N} + & \eta_0(LA)_{-N+1} + & \cdots & + \eta_{-N}(LA)_1 + 0 & & \\ \vdots & \vdots & & & & \\ \eta_N(LA)_{-N} + & \eta_{N-1}(LA)_{-N+1} + & \cdots & \eta_0(LA)_0 + & \cdots & + \eta_{-N}(LA)_N \\ & & & \vdots & & \vdots \\ 0 & & & \eta_N(LA)_0 + & \cdots & + \eta_0(LA)_N \end{array} \tag{7}$$

A typical $(LA)_m$ term can be expanded to

$$(LA)_m = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2}\right) A_m(r). \tag{8}$$

Using a nonuniform grid in radial direction, i.e., using uniform spacing in y, where

$$y^2 \equiv r,$$
$$2y\, dy = dr,$$

we can rewrite (8) as

$$(LA)_m = \left(\frac{1}{4y^3}\frac{\partial}{\partial y} y \frac{\partial}{\partial y} - \frac{m^2}{y^4}\right) A_m(r). \tag{9}$$

In finite different form, (9) is

$$(LA)_m = \alpha_i A_{i+1,m} + \beta_{im} A_{i,m} + \gamma_i A_{i-1,m}, \tag{10}$$

where

$$\alpha_i = y_{i+1/2}/(4y_i^3 \Delta^2), \tag{11}$$

$$\beta_{im} = -\left(\frac{1}{2y_i^2 \Delta^2} + \frac{m^2}{y_i^4}\right), \tag{12}$$

$$\gamma_i = \frac{y_{i-1/2}}{4y_i^3 \Delta^2}, \tag{13}$$

$$\Delta = y_{i+1} - y_i. \tag{14}$$

Substitute (10) in (7), and we obtain for the first row

$$\begin{aligned}
&\eta_0(\alpha_i A_{i+1,-N} + \beta_{i,-N} A_{i,-N} + \gamma_i A_{i-1,-N}) \\
&\quad + \eta_{-1}(\alpha_i A_{i+1,-N+1} + \beta_{i,-N+1} A_{i,-N+1} + \gamma_i A_{i-1,-N+1}) + \cdots \\
&\quad + \eta_{-N}(\alpha_i A_{i+1,0} + \beta_{i,0} A_{i,0} + \gamma_i A_{i-1,0}) \\
&= \alpha_i(\eta_0, \eta_{-1} \cdots \eta_{-N}) \begin{bmatrix} A_{-N} \\ A_{-N+1} \\ \vdots \\ A_0 \end{bmatrix}_{i+1} \\
&\quad + (\eta_0 \beta_{i,-N}, \eta_{-1}\beta_{i,-N+1} \cdots \eta_{-N}\beta_{i,0}) \begin{bmatrix} A_{-N} \\ A_{-N+1} \\ \vdots \\ A_0 \end{bmatrix}_{i} \\
&\quad + \gamma_i(\eta_0 \cdots \eta_{-N}) \begin{bmatrix} A_{-N} \\ A_{-N+1} \\ \vdots \\ A_0 \end{bmatrix}_{i-1} .
\end{aligned} \tag{15}$$

Similarly we can rewrite all the rows in (7) in a form such as (15). Consequently, the RHS of (5) can be expressed as

$$\bar{R}_i \bar{A}_{i+1} + \bar{S}_i \bar{A}_i + \bar{T}_i \bar{A}_{i-1}, \tag{16}$$

where $\bar{A}_{i+1}, \bar{A}_i, \bar{A}_{i-1}$ are column matrices with the first element equal to A_{-N} and the last element equal to A_N, and $\bar{R}_i$, $\bar{S}_i$, $\bar{T}_i$ are banded matrices with

$$\bar{R}_i = \bar{\alpha}_i \bar{\eta}_i, \tag{17}$$

$$\bar{S}_i = \bar{\eta}_i \bar{\beta}_i, \tag{18}$$

$$\bar{T}_i = \bar{\gamma}_i \bar{\eta}_i, \tag{19}$$

$$\bar{\eta}_i = \begin{bmatrix}
\eta_0 & \eta_{-1} & \cdots & & \eta_{-N} & & & \\
\eta_1 & \eta_0 & \eta_{-1} & \cdots & & & 0 & \\
\eta_2 & \eta_1 & \eta_0 & \cdots & & & & \\
\vdots & & & & & & & \\
\eta_N & \cdots & & & \eta_0 & \cdots & & \eta_{-N} \\
 & & & & \vdots & & & \vdots \\
0 & & & \eta_N & \eta_{N-1} & \cdots & \eta_0 & \eta_{-1} \\
 & & & & \eta_N & \cdots & \eta_1 & \eta_0
\end{bmatrix},$$

and $\bar{\alpha}_i$ and $\bar{\gamma}_i$ are diagonal matrices with their elements equal to α_i and γ_i, respectively. $\bar{\beta}_i$ is also a diagonal matrix, but each element is different, i.e.,

$$\bar{\beta}_i = \begin{bmatrix} \beta_{i,-N} & & 0 \\ & \ddots & \\ & & \beta_{i,N} \\ 0 & & \end{bmatrix}. \tag{20}$$

Going back to (5), we finally obtain

$$\bar{A}_i^{n+1} - \bar{A}_i^n = \frac{\delta}{2}\bar{R}_i\bar{A}_{i+1}^{n+1} + \bar{S}_i\bar{A}_i^{n+1} + \bar{T}_i\bar{A}_{i-1}^{n+1} + \frac{\delta}{2}\bar{R}_i\bar{A}_{i+1}^n + \bar{S}_i\bar{A}_i^n + \bar{T}_i\bar{A}_{i-1}^n, \tag{21}$$

$$\delta \equiv \zeta^{n+1} - \zeta^n. \tag{22}$$

The scheme described in (21) is similar to a Crank–Nicholson method and is $0(\delta)^2$ and $0(\Delta)^2$. Now we are ready to tackle the Lee field equations, as discussed in the next section.

IV. Implicit Formulation of the Lee Field Equations by Fourier Transform

We can cast Lee's field equations [3] into the form

$$\frac{\partial A}{\partial \zeta} = \eta\nabla_\perp^2(A + \phi) + \eta J_b, \tag{23}$$

$$\nabla_\perp^2\frac{\partial A}{\partial \zeta} = (\nabla\sigma)\cdot(\nabla\phi) + \sigma\nabla_\perp^2\phi, \tag{24}$$

where $\eta \equiv \sigma^{-1}$. Our aim here is to reduce (23) and (24) to a bi-tridiagonal system. The conductivity equation in normalized unit is

$$\frac{\partial\sigma}{\partial\zeta} = K_1 J_b + v_i\sigma - K_2 P\sigma^2,$$

where

$$K_1 = 1.4653,$$

$$K_2 = 1.7 \times 10^{-5},$$

$$v_i = \frac{APS^3}{1 + BS + CS^2 + DS^3},$$

$$S = E^2/P^2,$$

$$A = 1.423 \times 10^{-4},$$

$$B = 9.179 \times 10^{-6},$$

$$C = 2.656 \times 10^{-10},$$

$$D = 2.820 \times 10^{-17}.$$

E and P are the local electric field and pressure, respectively. The equation for σ or η is not solved implicitly because it includes nonlinear terms due to avalanche and recombination, but we can always time-center the σ equation to ensure accuracy to $0(\delta)^2$. For simplicity, let us define

$$\sigma' \equiv \frac{\partial \sigma}{\partial r}, \tag{25}$$

$$\sigma'' \equiv \frac{\partial \sigma}{\partial \theta}. \tag{26}$$

Equation (24) can be rewritten as

$$\nabla_\perp^2 \frac{\partial A}{\partial \zeta} = \sigma' \frac{\partial \phi}{\partial r} + \frac{\sigma''}{r^2} \frac{\partial \phi}{\partial \theta} + \sigma \nabla^2 \phi. \tag{27}$$

We shall now transform (23) and (27) term by term. From (23), we have

$$\frac{\partial A}{\partial \zeta} = (\bar{A}_i^{n+1} - \bar{A}_i^n)\, \delta^{-1}, \tag{28}$$

$$\eta \nabla_\perp^2 A = \tfrac{1}{2}[\bar{R}_i \bar{A}_{i+1}^{n+1} + \bar{S}_i \bar{A}_i^{n+1} + \bar{T}_i \bar{A}_{i-1}^{n+1}] + \tfrac{1}{2}[\bar{R}_i \bar{A}_{i+1}^n + \bar{S}_i \bar{A}_i^n + \bar{T}_i \bar{A}_{i-1}^n], \tag{29}$$

and

$$\eta \nabla_\perp^2 \phi = \tfrac{1}{2}[\bar{R}_i \bar{\phi}_{i+1}^{n+1} + \bar{S}_i \bar{\phi}_i^{n+1} + \bar{T}_i \bar{\phi}_{i-1}^{n+1}] + \tfrac{1}{2}[\bar{R}_i \bar{\phi}_{i+1}^n + \bar{S}_i \bar{\phi}_i^n + \bar{T}_i \bar{\phi}_{i-1}^n]. \tag{30}$$

The term $(\eta J_b)^{n+1/2}$ is calculated explicitly. From (27), we have

$$\nabla_\perp^2 \frac{\partial A}{\partial \zeta} = \delta^{-1}\{[\bar{\alpha}_i \bar{A}_{i+1}^{n+1} + \bar{\beta}_i \bar{A}_i^{n+1} + \bar{\gamma}_i \bar{A}_{i-1}^{n+1}] - [\bar{\alpha}_i \bar{A}_{i+1}^n + \bar{\beta}_i \bar{A}_i^n + \bar{\gamma}_i \bar{A}_{i-1}^n]\}, \tag{31}$$

$$\sigma' \frac{\partial \phi}{\partial r} = (8 y_i \Delta)^{-1}\, \bar{\sigma}_i'[(\bar{\phi}_{i+1} - \bar{\phi}_{i-1})^{n+1} + (\bar{\phi}_{i+1} - \bar{\phi}_{i-1})^n], \tag{32}$$

and

$$\frac{\sigma''}{r^2}\frac{\partial}{\partial\theta}\phi = -\frac{\bar{\imath}}{2y_i^4}\bar{\sigma}_i''\bar{m}[\bar{\phi}_i^{n+1}+\bar{\phi}_i^n], \tag{33}$$

where $\bar{\imath}\equiv(-1)^{1/2}$ to avoid confusion with index i. Also

$$\sigma\nabla_\perp^2\phi = \tfrac{1}{2}[\bar{R}_i'\bar{\phi}_{i+1}^{n+1}+\bar{S}_i'\bar{\phi}_i^{n+1}+\bar{T}_i'\bar{\phi}_{i-1}^{n+1}]+\tfrac{1}{2}[\bar{R}_i'\bar{\phi}_{i+1}^{n}+\bar{S}_i'\bar{\phi}_i^{n}+\bar{T}_i'\bar{\phi}_{i-1}^{n}]. \tag{34}$$

In the above, $\bar{m}$ is a diagonal matrix with element runs from $-N$ to N, and $\bar{R}_i'$, $\bar{S}_i'$ and $\bar{T}_i'$ are defined in (17)–(19) with $\bar{\eta}_i$ replaced by $\bar{\sigma}_i$. Putting (28)–(30) in (23) and (31)–(34) in (27), we get the desired bi-tridiagonal form:

$$a_i^{(1)}\bar{A}_{i-1}+a_i^{(2)}\bar{\phi}_{i-1}+b_i^{(1)}\bar{A}_i+b_i^{(2)}\bar{\phi}_i+c_i^{(1)}\bar{A}_{i+1}+c_i^{(2)}\bar{\phi}_{i+1}=d_i^{(1)}, \tag{35}$$

$$a_i^{(3)}\bar{A}_{i-1}+a_i^{(4)}\bar{\phi}_{i-1}+b_i^{(3)}\bar{A}_i+b_i^{(4)}\bar{\phi}_i+c_i^{(3)}\bar{A}_{i+1}+c_i^{(4)}\bar{\phi}_{i+1}=d_i^{(2)}, \tag{36}$$

where

$$a_i^{(1)} = -\frac{\delta\bar{T}_i}{2}, \tag{37}$$

$$a_i^{(2)} = -\frac{\delta\bar{T}_i}{2}, \tag{38}$$

$$a_i^{(3)} = \bar{\gamma}_i, \tag{39}$$

$$a_i^{(4)} = \frac{\delta\bar{\sigma}_i'}{8y_i\Delta}-\frac{\delta\bar{T}_i'}{2}, \tag{40}$$

$$b_i^{(1)} = \bar{I}-\frac{\delta\bar{S}_i}{2}, \tag{41}$$

$$b_i^{(2)} = -\frac{\delta\bar{S}_i}{2}, \tag{42}$$

$$b_i^{(3)} = \bar{\beta}_i, \tag{43}$$

$$b_i^{(4)} = \frac{\bar{\imath}\delta}{2y_i^4}\bar{\sigma}_i''\bar{m}-\frac{\delta\bar{S}_i'}{2}, \tag{44}$$

$$c_i^{(1)} = -\frac{\delta}{2}\bar{R}_i, \tag{45}$$

$$c_i^{(2)} = -\frac{\delta}{2}\bar{R}_i, \tag{46}$$

$$c_i^{(3)} = \bar{\alpha}_i, \tag{47}$$

$$c_i^{(4)} = -\left|\frac{\delta\bar{\sigma}_i'}{8y_i\Delta}+\frac{\delta\bar{R}_i'}{2}\right|, \tag{48}$$

$$d_i^{(1)} = \frac{\delta}{2}\bar{R}_i A_{i+1}^n + \left(I + \frac{\delta \bar{S}_i}{2}\right) A_i^n + \frac{\delta \bar{T}_i}{2} A_{i-1}^n$$

$$+ \frac{\delta}{2}(\bar{R}_i \phi_{i+1}^n + \bar{S}_i \phi_i^n + \bar{T}_i \phi_{i-1}^n) + (\eta J_b)^{n+1/2}\,\delta, \tag{49}$$

$$d_i^{(2)} = \bar{\alpha}_i \bar{A}_{i+1}^n + \bar{\beta}_i \bar{A}_i^n + \bar{\gamma}_i \bar{A}_{i-1}^n + \left(\frac{\delta \bar{\sigma}_i'}{8 y_i \Delta} + \frac{\delta \bar{R}_i'}{2}\right) \bar{\phi}_{i+1}^n$$

$$- \left(\frac{i\delta}{2y_i^4}\bar{\sigma}_i'' \bar{m} - \frac{\delta \bar{S}_i'}{2}\right) \bar{\phi}_i^n - \left(\frac{\delta \bar{\sigma}_i'}{8 y_i \Delta} - \frac{\delta \bar{T}_i'}{2}\right) \bar{\phi}_{i-1}^n, \tag{50}$$

where $\bar{I}$ is a unit diagonal matrix. The inversion method of the bi-tridiagonal system is given in Appendix A.

We use conducting boundary conditions at the outer boundary, i.e.,

$$\begin{aligned} A_m(R) &= 0, \\ \phi_m(R) &= 0, \end{aligned} \tag{51}$$

for all m. Near the origin, we require both the electric and magnetic fields be finite and continuous. Noting that for small r,

$$A_m(r), \phi_m(r) \propto r^m, \tag{52}$$

we have at once

$$A_m(0) = 0 = \phi_m(0), \qquad \text{for} \quad |m| \geqslant 1, \tag{53}$$

$$\frac{\partial A_m(0)}{\partial r} = 0 = \frac{\partial \phi_m(0)}{\partial r}, \qquad \text{for} \quad m = 0. \tag{54}$$

V. Applications

To date, this code has been used only to calculate beam equilibria, using a simple envelope model of the beam. The beam current density is specified to be of the form

$$J_b(r, \zeta) = f\left[\frac{|\mathbf{r} - Y(\zeta)|}{a(\zeta)}\right],$$

where f is a specified radial profile shape (usually Bennett), $a(\zeta)$ is the beam radius and $Y(\zeta)$ is the transverse displacement of the slice. In most cases, we have specified $a(\zeta)$ and used the code, iterating back in ζ from the beam head, to calculate the equilibrium value of $Y(\zeta)$. We have already mentioned briefly some applications of the code in Section I. We shall elaborate on these areas in some detail here.

Previous work [6] has shown that in the electrostatic regime at the beam head if

the beam radius r_b is smaller than the conductivity channel r_c, the beam is attracted towards the channel axis. On the other hand, if $r_b \geqslant r_c$ as is usually true in the beam head, we have shown that when the beam residues on the channel axis, each slice will see an average force repelling it from the channel. But in the electrostatic regime, if the beam is displaced sufficiently from the channel it should see an electric dipole attracting it back toward the channel. Using our code to study this problem, we found that when the beam displacement is of the order of the beam radius and $r_b \geqslant r_c$, a stable equilibrium exists with the beam off axis, i.e., $Y \neq 0$. The equilibrium displacement from the channel axis increases from the head of the beam to its tail. A detailed description of these results will be presented in a later report.

When an external electric or magnetic force F acts on the beam, each beam slice will be subject to a different acceleration $F/m\gamma$ if γ is a function of ζ. If the shear in $F(\zeta)/m\gamma(\zeta)$ is strong compared to the restoring force due to the beam's self-pinch, the beam will tear. The breakup will continue along ζ until for some value ζ_c the restoring force becomes stronger than the sheared external force; from this point on, i.e., $\zeta > \zeta_c$, the beam will hold together. Once the "guiding point" ζ_c is determined, we can find (a) the deflection due to the force F, (b) what portion of the beam is torn out and (c) the value of $\gamma(\zeta_c)$, i.e., the energy of that part of the beam actually guides the rest of the beam. If $\gamma(\zeta_c)$ differs substantially from γ ($\zeta = 0$), then a problem may occur in aiming the beam subject to the influence of this external force. This work is still in progress and the results will be reported in a later report.

APPENDIX A

The algorithm for inverting a bi-triagonal system with scalar coefficients is well known [7]. We shall generalize it to include matrix coefficients. The equations are

$$a_i^{(1)}u_{i-1} + a_i^{(2)}v_{i-1} + b_i^{(1)}u_i + b_i^{(2)}v_i + c_i^{(1)}u_{i+1} + c_i^{(2)}v_{i+1} = d_i^{(1)}$$

and

$$a_i^{(3)}u_{i-1} + a_i^{(4)}v_{i-1} + b_i^{(3)}u_i + b_i^{(4)}v_i + c_i^{(3)}u_{i+1} + c_i^{(4)}v_{i+1} = d_i^{(1)},$$

for $1 \leqslant i \leqslant R$, with $a_i^{(m)} = c_R^{(m)} = 0$ for $1 \leqslant m \leqslant 4$. The algorithm is as follows: First compute

$$\beta_i^{(1)} = b_i^{(1)} - a_i^{(1)}\lambda_{i-1}^{(1)} - a_i^{(2)}\lambda_{i-1}^{(3)},$$
$$\beta_i^{(2)} = b_i^{(2)} - a_i^{(1)}\lambda_{i-1}^{(2)} - a_i^{(2)}\lambda_{i-1}^{(4)},$$
$$\beta_i^{(3)} = b_i^{(3)} - a_i^{(3)}\lambda_{i-1}^{(1)} - a_i^{(4)}\lambda_{i-1}^{(3)},$$
$$\beta_i^{(4)} = b_i^{(4)} - a_i^{(3)}\lambda_{i-1}^{(2)} - a_i^{(4)}\lambda_{i-1}^{(4)},$$

with $\beta_1^{(m)} = b_1^{(m)}$ for $1 \leqslant m \leqslant 4$, and

$$\delta_i^{(1)} = d_i^{(1)} - a_i^{(1)}\gamma_{i-1}^{(1)} - a_i^{(2)}\gamma_{i-1}^{(2)},$$
$$\delta_i^{(2)} = d_i^{(2)} - a_i^{(3)}\gamma_{i-1}^{(1)} - a_i^{(4)}\gamma_{i-1}^{(2)},$$

with $\delta_i^{(1)} = d_1^{(1)}$ and $\delta_1^{(2)} = d_1^{(2)}$, and

$$\mu_i^{(1)} = \beta_i^{(2)^{-1}}\beta_i^{(1)} - \beta_i^{(4)^{-1}}\beta_i^{(3)},$$
$$\mu_i^{(2)} = \beta_i^{(1)^{-1}}\beta_i^{(2)} - \beta_i^{(3)^{-1}}\beta_i^{(4)},$$
$$\lambda_i^{(1)} = \mu_i^{(1)^{-1}}(\beta_i^{(2)^{-1}}c_i^{(1)} - \beta_i^{(4)^{-1}}c_i^{(3)}),$$
$$\lambda_i^{(2)} = \mu_i^{(1)^{-1}}(\beta_i^{(2)^{-1}}c_i^{(2)} - \beta_i^{(4)^{-1}}c_i^{(4)}),$$
$$\lambda_i^{(3)} = \mu_i^{(2)^{-1}}(\beta_i^{(1)^{-1}}c_i^{(1)} - \beta_i^{(3)^{-1}}c_i^{(3)}),$$
$$\lambda_i^{(4)} = \mu_i^{(2)^{-1}}(\beta_i^{(1)^{-1}}c_i^{(2)} - \beta^{(3)}{}_i^{-1}c_i^{(4)}),$$
$$\gamma_i^{(1)} = \mu_i^{(1)^{-1}}(\beta_i^{(2)^{-1}}\delta_i^{(1)} - \beta_i^{(4)^{-1}}\delta_i^{(2)}),$$
$$\gamma_i^{(2)} = \mu_i^{(2)^{-1}}(\beta_i^{(1)^{-1}}\delta_i^{(1)} - \beta_i^{(3)^{-1}}\delta_i^{(2)}).$$

Using

$$u_R = \gamma_R^{(1)},$$
$$v_R = \gamma_R^{(2)},$$

we get

$$\mu_i = \gamma_i^{(1)} - \lambda_i^{(1)}u_{i+1} - \lambda_i^{(2)}v_{i+1},$$
$$v_i = \gamma_i^{(2)} - \lambda_i^{(3)}u_{i+1} - \lambda_i^{(4)}v_{i+1},$$

for $(R-1) \geqslant i \geqslant 1$.

Appendix B

For the sake of accuracy, we want to include as many modes as possible, but the running time goes up as the cube of the number of modes, i.e., if we double the number of modes, it will take eight times longer to run. To monitor the accuracy, one should make sure that the absolute values of the transforms decrease monotonically from the lowest mode to the highest mode. Moreover, the absolute value of the lowest mode should be at least two orders of magnitude larger than the highest mode. Depending on the symmetry of the problem being studied, it usually takes eight modes to study a beam displacement of $Y/a \sim 1$. As far as the field solver is concerned, because it is solved implicitly, one can use relatively large time steps ($\Delta\zeta$). But the conductivity equation is nonlinear and is solved explicitly, and one can

usually avoid difficulties by choosing $\Delta\zeta/a \sim 0.5$. The y step size is usually 0.02 beam radius.

The source term in J_b can be externally imposed, i.e., assigning a specific $J_b(\zeta)$, for example, for a Bennett beam one could use

$$J_b(\zeta) = \frac{I_b(\zeta)}{\pi a^2} \frac{1}{(1 + r^2/a^2)^2},$$

where $I_b(\zeta)$ is the beam current with some kind of a rise time. Alternatively, if this code is joined with a particle pusher, then J_b is obtained by summing up the beam charges. J_{bm} is obtained by transforming J_b. A typical run using four modes ($m = 8$), 200 y-grids and 60 ζ steps takes about 40 minutes of VAX 11/780 CPU time.

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REFERENCES

1. G. JOYCE AND M. LAMPE, "Proceedings, Tenth Conference on Numerical Simulation of Plasmas, San Diego, 1983"; F. W. CHAMBERS, "Mathematical Methods for the Ringbearer Simulation Code," Lawrence Livermore National Laboratory Report UCID-19494, 1982.
2. E. P. LEE, F. W. CHAMBERS, L. L. LODESTRO, AND S. S. YU, Stable propagation of an electron beam in gas, *in* "Proceedings, Second International Conference on High Power Electron and Ion Beam Research and Technology, Cornell University, 1977," p. 381; R. F. HUBBARD, M. LAMPE, S. SLINKER, AND G. JOYCE, "Proceedings, Tenth Conference on Numerical Simulation of Plasmas, San Diego, 1983"; "VIPER I-A Multi-Component Hose Dynamics Code," JAYCOR Technical Report J207-81-005, 1981.
3. E. P. LEE, "The New Field Equations," Lawrence Livermore National Laboratory Report UCID-17826, 1976.
4. B. HUI AND M. LAMPE, Naval Research Laboratory, Memorandum Report 5136, 1983.
5. D. GOTLIEB AND S. ORSZAG, "Numerical Analysis of Spectral Methods: Theory and Applications," CBMS–NSF Regional Conference Series in Applied Mathematics, SIAM, 1977.
6. J. MASAMITSU, S. YU, AND F. CHAMBERS, "Beam Tracking Studies with Ringbearer II," Lawrence Livermore National Laboratory Report UCID-19771, 1982.
7. D. V. ROSENBURG, "Method for the Numerical Solution of Partial Differential Equation" (R. Bellman, Ed.), Elsevier, Amsterdam/New York, 1969.